

Charm and Bottom Masses from Sum Rules
with a Convergence TestBAHMAN DEHNADI ^{*,1}, ANDRÉ H. HOANG ^{*,†}, VICENT MATEU ^{*}^{*} *University of Vienna, Faculty of Physics, Boltzmannngasse 5, A-1090 Wien, Austria*[†] *Erwin Schrödinger International Institute for Mathematical Physics, University of Vienna, Boltzmannngasse 9, A-1090 Vienna, Austria*

In this talk we discuss results of a new extraction of the $\overline{\text{MS}}$ charm quark mass using relativistic QCD sum rules at $\mathcal{O}(\alpha_s^3)$ based on moments of the vector and the pseudoscalar current correlators and using the available experimental measurements from e^+e^- collisions and lattice results, respectively. The analysis of the perturbative uncertainties is based on different implementations of the perturbative series and on independent variations of the renormalization scales for the mass and the strong coupling following a work we carried out earlier. Accounting for the perturbative series that result from this double scale variation is crucial since some of the series can exhibit extraordinarily small scale dependence, if the two scales are set equal. The new aspect of the work reported here addresses the problem that double scale variation might also lead to an overestimate of the perturbative uncertainties. We supplement the analysis by a convergence test that allows to quantify the overall convergence of QCD perturbation theory for each moment and to discard series that are artificially spoiled by specific choices of the renormalization scales. We also apply the new method to an extraction of the $\overline{\text{MS}}$ bottom quark mass using experimental moments that account for a modeling uncertainty associated to the continuum region where no experimental data is available. We obtain $\overline{m}_c(\overline{m}_c) = 1.287 \pm 0.020 \text{ GeV}$ and $\overline{m}_b(\overline{m}_b) = 4.167 \pm 0.023 \text{ GeV}$.

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Introduction: Precise and reliable determinations of the charm and bottom quark masses are an important input for a number of theoretical predictions such as Higgs branching ratios to charm and bottom quarks or for the corresponding Yukawa couplings [1]. They also affect the theoretical predictions of radiative and inclusive B decays, as well as rare kaon decays. For example, the inclusive semileptonic decay rate of B mesons depends on the fifth power of the bottom quark mass. These weak decays provide crucial methods to determine elements of the CKM matrix and which in turn are important for testing the validity of the Standard Model and for indirect searches of new physics. In this context having a reliable estimate of uncertainties of the quark masses is as important as knowing their values precisely [2]. Due to confinement quark masses are not physical observables. Rather, they are scheme-dependent parameters of the QCD Lagrangian which have to be determined from quantities that strongly depend on them.

One of the most precise tools to determine the charm and bottom quark masses is the QCD sum rule method where weighted averages of the normalized cross section $R_{e^+e^- \rightarrow q\bar{q}+X}$, with $q = c, b$, can be related to moments of the quark vector current correlator Π_V which can be calculated in perturbation theory using the OPE:

$$M_n = \int \frac{ds}{s^{n+1}} R_{e^+e^- \rightarrow q\bar{q}+X}(s), \quad R_{e^+e^- \rightarrow c\bar{c}+X}(s) = \frac{\sigma_{e^+e^- \rightarrow c\bar{c}+X}(s)}{\sigma_{e^+e^- \rightarrow \mu^+\mu^-}(s)}, \quad (1)$$

$$M_n^{\text{th}} = \frac{12\pi^2 Q_q^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_V(q^2) \Big|_{q^2=0}, \quad j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x),$$

$$(g_{\mu\nu} q^2 - q_\mu q_\nu) \Pi_V(q^2) = -i \int dx e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle.$$

Here Q_q is the quark electric charge and $\sqrt{s} = \sqrt{q^2}$ is the e^+e^- center of mass energy. Since the integration over the experimental R-ratio stretches from the quark pair threshold up to infinity and since useful experimental measurements only exist for energies up to around 11 GeV, the method relies on using theory input for energies above that scale (which we call “continuum”). For the charm case the combination of all available measurements is sufficient to render the experimental moments independent of uncertainties one assigns to the theory input for the continuum region [3]. For the bottom case the dependence on the continuum theory input is very large, and the dependence of the low-n experimental moments on unavoidable assumptions about the continuum error can be the most important component of the error budget. In fact, the use of the first moment M_1 appears to be excluded until experimental data become available for higher energies.

Alternatively one can also consider moments of the pseudoscalar current correlator. Experimental information on the correlator Π_P is not available in a form useful for quark mass determinations, but for charm quarks very precise lattice calculations have become available recently [8]. For the pseudo-scalar correlator it turns out that

the first two Taylor coefficients in the small- q^2 expansion need to be renormalized and that the third term (which we will denote by M_0^P) is hardly sensitive to m_q . We adopt the definitions

$$\begin{aligned}\Pi_P(q^2) &= i \int dx e^{iqx} \langle 0 | T j_P(x) j_P(0) | 0 \rangle, \quad j_P(x) = 2 m_q i \bar{\psi}(x) \gamma_5 \psi(x), \\ M_n^P &= \left[\frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n P(q^2) \Big|_{q^2=0} \right]^{\frac{1}{2n}}, \quad P(q^2) = \frac{\Pi_P(q^2) - \Pi_P(0) - \Pi'_P(0) q^2}{q^4},\end{aligned}\tag{2}$$

where the explicit mass factor in the definition of the pseudo-scalar current ensures that it is renormalization scheme independent. For the vector as well as the pseudoscalar correlator moments at low values of n , it is mandatory to employ a short-distance mass scheme such as $\overline{\text{MS}}$, which renders the quark mass $\overline{m}_q(\mu_m)$ dependent on its renormalization scale μ_m , similar to the strong coupling $\alpha_s(\mu_\alpha)$ which depends on μ_α .

Previous results: In a number of recent low- n sum rule analyses [4, 5, 6, 7, 8, 9, 10] based on $\mathcal{O}(\alpha_s^3)$ perturbation theory [11, 12, 13] the constraint $\mu_m = \mu_\alpha$ was employed. In Ref. [3] we analyzed the perturbative series for the vector correlator moments $M_{1,2,3,4}$ at $\mathcal{O}(\alpha_s^3)$ with the aim to reliably assess their uncertainty coming from the truncation of the series and to find out whether renormalization scale variation with $\mu_m = \mu_\alpha$ gives a compatible estimate of this uncertainty. We considered four different alternatives, each of which providing a viable analytic expression to carry out the charm mass determination:

1. conventional **fixed order** expansion for M_n and solving for $\overline{m}_q(\mu_m)$ numerically;
2. taking the $2n$ -th root of M_n and expanding in $\alpha_s(\mu_\alpha)$, which achieves a **linearized** relation of the inverse of the root of the experimental moment to \overline{m}_q ;
3. solving for the mass in a **linearized iterative** way as a function of the experimental moment and $\alpha_s(\mu_\alpha)$;
4. adopting an energy-dependent renormalization scale for α_s of the form $\mu_\alpha^2 \rightarrow \mu_\alpha^2 [1 - q^2/4\overline{m}_q(\mu_m)^2]$ and Taylor-expanding Π_V around $q^2 = 0$.

The fourth method is related to a **contour improved** dispersive integral for the moments in the complex q^2 plane, and the four expansion methods can also be used for the pseudo-scalar correlator. The important observations made in Ref. [3] were:

- Using expansions 1-4 with variations of μ_m and μ_α that are correlated (such as $\mu_m = \mu_\alpha$) can lead to charm mass results with very small error estimates that are not compatible with each other. For some expansions it happens that the results from the different orders are incompatible to each other. This indicates that correlated scale variation in general underestimates the perturbative uncertainty.

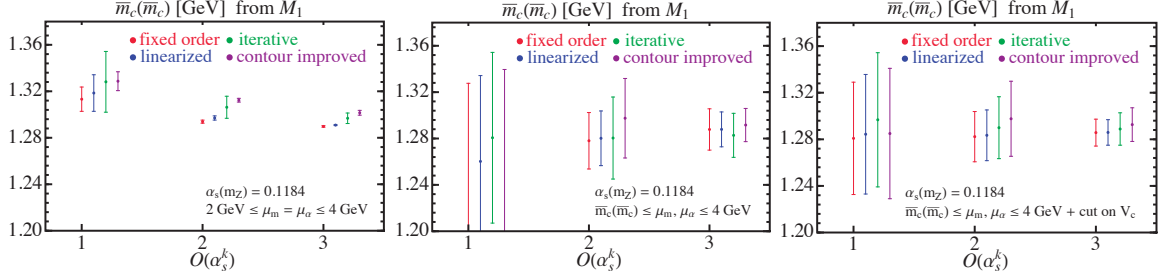


Figure 1: Charm mass values plus scale variation from the first moment of the vector current at $\mathcal{O}(\alpha_s^{1,2,3})$, for expansions 1-4; (a) correlated scale variation between 2 GeV and 4 GeV (b) uncorrelated scale variation between $\overline{m}_c(\overline{m}_c)$ and 4 GeV, and (c), as in (b) discarding 3% of series with highest V_c values.

- Uncorrelated, i.e. independent variation of μ_m and μ_α leads to charm mass results with error estimates that are in general larger, but also fully compatible among the expansions 1-4 and among the different orders.
- The size of the perturbative error for the charm mass depends quite strongly on the choice of the lower bound of the range of the scale variation. The choice of the upper bound only has a relatively small impact.

Figure 1(a) shows the results for $\overline{m}_c(\overline{m}_c)$ for expansions 1-4 and at $\mathcal{O}(\alpha_s^{1,2,3})$ for $2 \text{ GeV} \leq \mu_m = \mu_\alpha \leq 4 \text{ GeV}$. For example, in Ref. [7] and in Ref. [6] the fixed order and the linearized expansions, respectively, were employed using this type of correlated scale variation, leading to a perturbative error estimate at the level of 1 to 2 MeV. Given these observations we concluded that uncorrelated variations of μ_m and μ_α have to be used for a reliable estimate of the perturbative uncertainty. We further argued that the proper range of variation should include the charm mass itself, the scale that governs the series, and we adopted the range $\overline{m}_c(\overline{m}_c) \leq \mu_m, \mu_\alpha \leq 4 \text{ GeV}$, which is motivated by the range $2 m_c \pm m_c$ around the pair production threshold. The outcome is shown in Fig. 1(b) giving a perturbative error estimate of around 20 MeV. The final result quoted in Ref. [3] for $\alpha_s(m_Z) = 0.1184 \pm 0.0021$ was $\overline{m}_c(\overline{m}_c) = 1.282 \pm (0.006)_{\text{stat}} \pm (0.009)_{\text{syst}} \pm (0.019)_{\text{pert}} \pm (0.010)_{\alpha_s} \pm (0.002)_{\langle GG \rangle} \text{ GeV}$ based on the expansion 3 (linearized iterative method).

New results with a convergence test: While certainly being a conservative method, one point of concern in using an uncorrelated scale variation is that it might lead to an overestimate of the perturbative error, e.g. due to logarithms $\ln(\mu_m/\mu_\alpha)$ in connection with the low-scale strong coupling $\alpha_s(\mu_\alpha)$ that might artificially spoil the perturbative series. One approach might be to simply reduce the range of scale variation, in particular the lower bound; but this does not resolve the issue since the resulting smaller variation then simply represents a matter of choice. Rather, the issue should be resolved from properties of the perturbative series themselves. In Ref. [14]

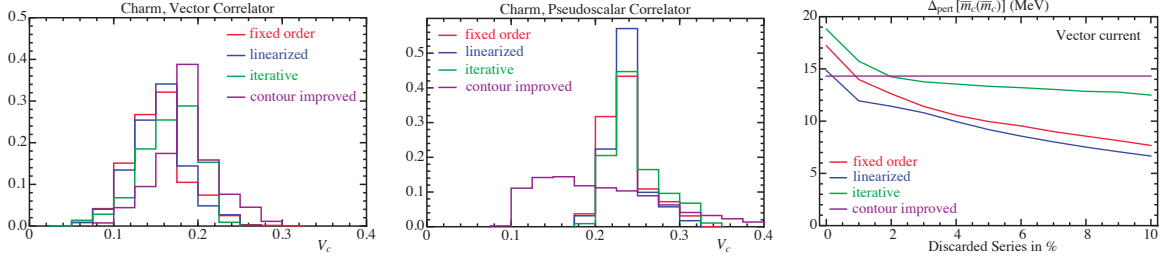


Figure 2: (a) V_c distribution for $\bar{m}_c(\bar{m}_c)$ from vector M_1 for expansions 1-4; (b) same from pseudoscalar M_1^P ; (c) half of the scale variation on $\bar{m}_c(\bar{m}_c)$ from M_1 at $\mathcal{O}(\alpha_s^3)$ as a function over the fraction of the discarded series with highest V_c values.

we address this issue by supplementing the uncorrelated scale variation method with a convergence test constraint, which we explain in the following and which leads to an updated result for the charm mass from the vector current moment analysis. In addition we use the new approach to analyze the charm quark pseudoscalar current moments as well as the vector current moments for bottom quarks. Due to lack of space in these proceedings we only present the main idea and some results and refer to Ref. [14] for details. The convergence test is as follows:

- (a) For each pair (μ_m, μ_α) the convergence parameter V_c is calculated from the charm mass series $\bar{m}_c(\bar{m}_c) = m^{(0)} + \delta m^{(1)} + \delta m^{(2)} + \delta m^{(3)}$ that results from the fits at $\mathcal{O}(\alpha_s^{0,1,2,3})$:

$$V_c = \max \left[\frac{\delta m^{(1)}}{m^{(0)}}, \left(\frac{\delta m^{(2)}}{m^{(0)}} \right)^{1/2}, \left(\frac{\delta m^{(3)}}{m^{(0)}} \right)^{1/3} \right]. \quad (3)$$

- (b) The resulting distribution of V_c values is a measure for the overall convergence of the perturbative expansion that is employed. If the distribution is peaked around the average $\langle V_c \rangle$ it has a well-defined convergence. Hence discarding series with $V_c \gg \langle V_c \rangle$ (which otherwise significantly enlarge the error) is justified.

The resulting V_c distributions of expansions (1, 2, 3, 4) for the analysis of the vector moment M_1 are shown in Fig. 2a, with $\langle V_c \rangle = (0.15, 0.15, 0.17, 0.19)$, clearly indicating a very good overall convergence. Figure 2c shows the scale variation error (= half the overall variation) as a function of the fraction of the series (with the largest V_c values) that are being discarded. We see that, indeed, only about 2% of the series with the highest V_c values are responsible for increasing the scale variation from well below 15 MeV to up to 20 MeV. For our final estimate we discard 3% of the series which leads to an updated result of (using $M_1^{\text{exp}} = 0.2121 \pm 0.0020_{\text{stat}} \pm 0.0030_{\text{syst}}$ [3])

$$\begin{aligned} \bar{m}_c(\bar{m}_c) = & 1.287 \pm (0.006)_{\text{stat}} \pm (0.009)_{\text{syst}} \pm (0.014)_{\text{pert}} \\ & \pm (0.010)_{\alpha_s} \pm (0.002)_{\langle GG \rangle} \text{ GeV} \end{aligned} \quad (4)$$

for the charm mass from the vector correlator analysis using the iterative expansion as the default. Fig. 1(c) shows the outcome for the other expansions. We also indicate in Eq. (4) the uncertainty from $\alpha_s(m_Z) = 0.1184 \pm 0.0021$ and the gluon condensate. This is the main result of our analysis for the charm mass. We also applied the same approach to extract the charm mass from the pseudo-scalar moment M_1^P . The resulting V_c distributions are shown in Fig. 2b and again show a clearly visible peak. However, with $\langle V_c \rangle = (0.24, 0.24, 0.25, 0.21)$, the average V_c values are clearly larger indicating that the pseudoscalar moment has a perturbative convergence that is worse than for the vector moment. This means that the vector correlator method is superior, and that it is expected that the perturbative uncertainty in the charm mass from the pseudo-scalar is larger. We find (using $M_1^{P,\text{latt}} = 0.5127 \pm 0.0037$ [8])

$$\overline{m}_c(\overline{m}_c) = 1.266 \pm (0.008)_{\text{lat}} \pm (0.035)_{\text{pert}} \pm (0.019)_{\alpha_s} \pm (0.002)_{\langle GG \rangle} \text{ GeV} \quad (5)$$

using again the iterative expansion as the default.

Using our method we can also determine the $\overline{\text{MS}}$ bottom quark mass from the vector correlator. For the determination of the experimental moments from the region above 11.2 GeV we use pQCD (which has essentially negligible errors) supplemented by a modeling uncertainty. Comparing pQCD and rebinned data in the region between 11.06 GeV and 11.2 GeV we find a 4% discrepancy. Given that the rel. discrepancy between experiment and pQCD for R_b at the Z-pole is about 3 permille [15], we adopt a rel. modeling error that decreases linearly from 4% at 11.2 GeV to 3 permille at m_Z , and which is the pQCD error above. This uncertainty makes up for 96.9% of the total error for the first moment M_1 (which has an overall 2.45% relative error), and 86.27% of the second moment M_2 (which has a overall 1.85% relative error). Note that if we would adopt a constant 4% error for all energies above 11.2 GeV, this continuum uncertainty would make up for 97.25% of the total error for the first moment M_1 (which has an overall 2.59% relative error), and 86.57% of the second moment M_2 (which has a overall 1.86% relative error). The difference is small because contributions from higher energies are suppressed. For more details on these considerations we refer to Ref. [14]. In our analysis we use the second moment M_2 and employ uncorrelated scale variations in the range $\overline{m}_b(\overline{m}_b) \leq \mu_m, \mu_\alpha \leq 15 \text{ GeV}$. Interestingly we find that the convergence test for M_2 gives $\langle V_c \rangle = (0.13, 0.11, 0.12, 0.15)$ for expansions (1, 2, 3, 4), and we further find that discarding series with the highest V_c values only has minor effects on the perturbative error estimate for fractions up to 5%. This indicates that the perturbative series for bottom moments are more stable, which is expected from the fact that perturbation theory should work better for the bottom than for the lighter charm. Our final result for the $\overline{\text{MS}}$ bottom quark mass reads discarding again 3% of the series with highest V_c values.

$$\begin{aligned} \overline{m}_b(\overline{m}_b) = & 4.167 \pm (0.004)_{\text{stat}} \pm (0.018)_{\text{syst}} \pm (0.010)_{\text{pert}} \\ & \pm (0.007)_{\alpha_s} \pm (0.001)_{\langle GG \rangle} \text{ GeV} \end{aligned} \quad (6)$$

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